

On Modelling a Relativistic Hierarchical (Fractal) Cosmology by Tolman's Spacetime . II. Analysis of the Einstein–de Sitter Model

Marcelo B. Ribeiro

Astronomy Unit
School of Mathematical Sciences
Queen Mary and Westfield College
Mile End Road
London E1 4NS
England

ABSTRACT

This paper studies the spatially homogeneous Einstein–de Sitter cosmological model in the context of a relativistic hierarchical (fractal) cosmology as developed in paper I. We treat the Einstein–de Sitter model as a special case of Tolman's spacetime, obtained by the appropriate choice of the latter's three arbitrary functions. We calculate the observational relations along the past light cone of the model under consideration and carry out an investigation of whether or not it has fractal behaviour. We have found that the Einstein–de Sitter model does not seem to remain homogeneous along the geodesic and that it also has no fractal features along the backward null cone.

Subject headings: cosmology: theory

1 Introduction

The recent galactic redshift surveys showing a very inhomogeneous picture for the distribution of galaxies (de Lapparent, Geller & Huchra 1986; Saunders et al. 1991) have stimulated the trend of study of the galactic clustering problem which assumes that the large scale structure of the universe can be described as being a self-similar fractal system. Under this philosophy there have been studies in which the distribution of galaxies is assumed to form a multifractal system (Jones et al. 1988; Martínez et al. 1990), but the first single fractal Newtonian model advanced as a description of the large scale structure is due to Pietronero (1987).

This paper is the second of a series where we develop a relativistic fractal cosmology obtained by the relativistic generalization of Pietronero's (1987) model. In paper I (Ribeiro 1992) we have argued that the all sky redshift surveys mentioned above present observations consistent both with the old Charlier's hypothesis of hierarchical clustering and with fractals, where the latter is basically a more precise conceptualization of the scaling idea implicit in the hierarchical clustering hypothesis. Also in paper I we have used Pietronero's basic hypothesis to propose similar ones in a relativistic framework and to develop observational relations compatible with fractals in Tolman's spacetime.

The purpose of this paper is to apply those observational relations to the specific and analytically manageable special case of the Einstein-de Sitter model, which is obtained by a particular choice of the three arbitrary functions of Tolman's spacetime. Our basic goal here is to answer the question of whether or not the Einstein-de Sitter model is compatible with a fractal description of galactic clustering as developed in paper I, and we intend to do so by studying the consequences of paper I's observational relations in a Einstein-de Sitter universe. We believe that this problem must be addressed because Einstein-de Sitter is spatially homogeneous and it is not at all clear what would be the behaviour of its fractal observational relations as we go down the geodesic, through hypersurfaces of constant t each having a different value for the proper density. The calculations shown next also serve as an illustration of the theory developed in paper I.

The plan of the paper is as follows. In §2 we describe very briefly the Tolman observational relations presented in paper I and discuss how the Tolman metric relates to its Friedmann counterpart. In §3 we apply these observational relations to the Einstein-de Sitter case and analyse the results. There is a conclusion in §4.

2 Observational Relations in Tolman's Spacetime

The Tolman (1934) metric for the motion of spherically symmetric dust is

$$dS^2 = dt^2 - \frac{R'^2}{f^2} dr^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (1)$$

The Einstein field equations (with $c = G = 1$ and $\Lambda = 0$) for metric (1) reduce to a single equation

$$2R\dot{R}^2 + 2R(1 - f^2) = F, \quad (2)$$

where a dot means $\partial/\partial t$ and a prime means $\partial/\partial r$; $f(r)$ and $F(r)$ are two arbitrary functions and $R(r, t) \geq 0$. For $f^2 = 1$ the field equation (2) has the following solution:

$$R = \frac{1}{2}(9F)^{1/3}(t + \beta)^{2/3} \quad (3)$$

where $\beta(r)$ is a third arbitrary function which gives the local time passed since the singularity surface, that is, since the big bang. Equation (2) also has solutions for $f^2 < 1$ and $f^2 > 1$, but they are of no interest for us here (see paper I and references therein). The local density is given by

$$8\pi\rho = \frac{F'}{2R'R^2} \quad (4)$$

and if we adopt the radius coordinate r as the parameter along the backward null cone, we can then write the radial null geodesic of metric (1) as

$$\frac{dt}{dr} = -\frac{R'}{f}. \quad (5)$$

For the sake of clarity of this work, we shall briefly describe the relationship of the equations above to the usual Friedmann spacetime. The junction conditions between the Tolman and Friedmann metrics calculated in paper I show that in order to obtain the latter from the former we have to assume that

$$R(r, t) = a(t) g(r), \quad f(r) = g'(r). \quad (6)$$

By substituting equations (6) into the metric (1) we get

$$dS^2 = dt^2 - a^2(t) \left\{ dr^2 - g^2(r) [d\theta^2 + \sin^2 \theta d\phi^2] \right\}, \quad (7)$$

which is a Friedmann metric if

$$g(r) = \begin{cases} \sin r, \\ r, \\ \sinh r. \end{cases} \quad (8)$$

If we now substitute equations (6) in equation (4) and integrate it, that gives

$$\frac{F}{4} = \frac{4\pi}{3} \rho a^3 g^3. \quad (9)$$

It is worth noting that the time derivative of the equation above gives the well-known relation for the matter-dominated era of a Friedmann universe:

$$\frac{d}{dt} (\rho a^3) = 0.$$

Equation (9) is necessary in order to show the physical role played by $F(r)$ and to deduce the usual Friedmann equation. This is possible by substituting equations (6) and (9) into equation (2). The result may be written as

$$\dot{a}^2 = \frac{8\pi}{3} \rho a^2 - K \quad (10)$$

where

$$K = \frac{1 - g'^2}{g^2}.$$

It is easy to see that $K = +1, 0, -1$ if $g = \sin r, r, \sinh r$, respectively, and this shows that equation (10) is indeed the usual Friedmann equation. Let us now write equation (10) in the form

$$\frac{\dot{a}^2 g^2}{2} - \frac{m}{ag} = - (1 - g'^2) \quad (11)$$

where

$$m(r) = \frac{4\pi}{3} \rho a^3 g^3. \quad (12)$$

Equation (11) is interpreted as an energy equation and, in consequence, $m(r)$ is the gravitational mass inside the coordinate r . Thus $4m(r) = F(r)$ and this clarifies the role of the function $F(r)$ in providing the gravitational mass of the system. In addition, equation (11) shows that the function $f(r)$ in Tolman's spacetime gives the total energy of the system (see also paper I, §4).

Finally, a word should be said about the function $\beta(r)$ that gives the big bang time. If “now” is defined as $t = 0$ and if $\beta(r) = 0$, then the hypersurface $t = 0$ is singular, that is, $R = 0$ everywhere¹. So $\beta(r)$ gives the age of the universe which in Tolman's spacetime may change if different observers are situated at different radial coordinates r . This is a remarkable departure from Friedmann's model that gives the same age of the universe for all observers on a hypersurface of constant t . In other words, in a Friedmann universe the big bang is simultaneous while in a Tolman one it

¹This is also valid for $f^2 > 1$ and $f^2 < 1$ type solutions of equation (2). See paper I, §3, for the analytical expressions of R in these two cases.

may be non-simultaneous, that is, the big bang may have occurred at different proper times in different locations. As a consequence another essential ingredient in reducing the Tolman metric to Friedmann is $\beta = \text{constant}$, and so the linkage between β and the Hubble constant is of the form

$$\frac{\dot{R}}{R} = \frac{\dot{a}}{a} = H(t), \quad \text{for } \beta = \beta_0, \quad (13)$$

where β_0 is a constant. Considering equation (3) it is straightforward to conclude that

$$\beta_0 = \frac{2}{3H_0}, \quad (14)$$

where $H_0 = H(0)$. Equation (14) gives the relationship between β_0 and the Hubble constant H_0 in a Einstein-de Sitter universe.

Let us now return to the discussion of the metric (1) and its observational relations. It was shown in paper I that in Tolman's spacetime the redshift may be written as

$$1 + z = (1 - I)^{-1} \quad (15)$$

where the function $I(r)$ is the solution of the differential equation

$$\frac{dI}{dr} = \frac{\dot{R}'}{f}(1 - I). \quad (16)$$

The luminosity distance d_l and the cumulative number count N_c are given by

$$d_l = R(1 + z)^2, \quad (17)$$

$$N_c = \frac{1}{4M_G} \int \frac{F'}{f} dr, \quad (18)$$

where M_G is the average galactic rest mass ($\sim 10^{11} M_\odot$). The volume V of the sphere which contains the sources, and the volume density (average density) ρ_v , have the form

$$V = \frac{4}{3}\pi(d_l)^3, \quad (19)$$

$$\rho_v = \frac{N_c M_G}{V}. \quad (20)$$

The proposed relativistic version of Pietronero's (1987) generalized mass-length relation is given by

$$N_c = \sigma(d_l)^D \quad (21)$$

where σ is a constant related to the lower cutoff of the fractal system and D is its fractal dimension. If we substitute equations (19) and (21) into equation (20) we get de Vaucouleurs' density power law

$$\rho_v = \frac{3\sigma M_G}{4\pi}(d_l)^{-\gamma}, \quad \gamma = 3 - D. \quad (22)$$

3 Application to the Einstein–de Sitter Case

As discussed in the previous section, the Einstein–de Sitter model can be obtained from Tolman’s spacetime as a special case when the three arbitrary functions take the form

$$f(r) = 1, \quad F(r) = \frac{8}{9}r^3, \quad \beta(r) = \beta_0, \quad (23)$$

where β_0 is a constant. The solution (3) of the field equation and its derivatives for this case may be seen below:

$$\begin{cases} R(r, t) = r(t + \beta_0)^{2/3}; & \dot{R}(r, t) = \frac{2}{3}r(t + \beta_0)^{-1/3}; \\ R'(r, t) = (t + \beta_0)^{2/3}; & \dot{R}'(r, t) = \frac{2}{3}(t + \beta_0)^{-1/3}. \end{cases} \quad (24)$$

The integration of the null geodesic (5) from $t = 0$, $r = 0$ till $t(r)$ results in

$$3(t + \beta_0)^{1/3} = 3\beta_0^{1/3} - r. \quad (25)$$

With the solution (25) we can then write equations (24) along the null geodesic:

$$\begin{cases} R = \frac{r}{9}(3\beta_0^{1/3} - r)^2; & \dot{R} = 2r(3\beta_0^{1/3} - r)^{-1}; \\ R' = \frac{1}{9}(3\beta_0^{1/3} - r)^2; & \dot{R}' = 2(3\beta_0^{1/3} - r)^{-1}. \end{cases} \quad (26)$$

Using the appropriate equation (26) and integrating equation (16) from $I = 0$, $r = 0$ to $I(r)$, it is straightforward to show that

$$1 - I = \left(\frac{3\beta_0^{1/3} - r}{3\beta_0^{1/3}} \right)^2, \quad (27)$$

and the redshift (15) parametrized along the geodesic becomes

$$1 + z = \left(\frac{3\beta_0^{1/3}}{3\beta_0^{1/3} - r} \right)^2. \quad (28)$$

If we consider equations (23) and (28), the luminosity distance, number counts, volume and volume density are easily found to be

$$d_l = \frac{9r\beta_0^{4/3}}{(3\beta_0^{1/3} - r)^2}, \quad (29)$$

$$N_c = \frac{2r^3}{9M_G}, \quad (30)$$

$$V = \frac{12\pi r^3(3\beta_0)^4}{(3\beta_0^{1/3} - r)^6}, \quad (31)$$

$$\rho_v = \frac{(3\beta_0^{1/3} - r)^6}{54\pi(3\beta_0)^4}. \quad (32)$$

As one can see the volume density here is expressed in terms of a parameter along the geodesic. Bonnor (1972) carried out a study bearing some similarities to the one presented in this paper, but with the major difference that here all observational relations are calculated along the backward null cone, since this is where astronomical observations are actually made. Bonnor's (1972) volume density only applies to the present time hypersurface. Equation (32) may change along the geodesic because it is a cumulative density, averaging at bigger and bigger volumes with different local densities.

Finally, the local density (4) is given by

$$\rho = \frac{1}{6\pi(t + \beta_0)^2}, \quad (33)$$

and along the geodesic this equation becomes

$$\rho = \frac{1}{6\pi} \left(\frac{3}{3\beta_0^{1/3} - r} \right)^6. \quad (34)$$

It is important to stress that the presence of the coordinate r in equation (34) does not mean a spatially inhomogeneous local density. The radial coordinate is only a parameter along the geodesic and, due to the past null geodesic equation (25), each value of r corresponds to a single value of t . In other words, each r corresponds to a specific $t = \text{constant}$ hypersurface given by equation (25) and, hence, the density in equation (34) is homogeneous, that is, constant at each such hypersurface. However, equation (34) effectively changes along the geodesic as it goes through different surfaces of t constant and in this sense the model can be thought of as inhomogeneous.

An interesting consequence follows immediately from the results above. If we use equation (28) and express ρ_v in terms of the redshift, we get

$$\rho_v = \frac{1}{6\pi\beta_0^2(1+z)^3} \quad (35)$$

which expanded in power series turns out to be

$$\rho_v = \frac{1}{6\pi\beta_0^2} (1 - 3z + 6z^2 - 10z^3 + \dots). \quad (36)$$

We can immediately see from equation (36) that for small redshifts the volume density is constant, but as soon as z increases ρ_v begins to depart from a constant value. Therefore, the spatially homogeneous Einstein-de Sitter model does not appear to

have constant volume density as we go down the geodesic. We shall return to this point later.

As the function $\beta(r)$ determines the local time at which $R = 0$, the surface $t + \beta = 0$ is a surface of singularity and, hence, the physical region considered is given by the condition $t + \beta > 0$. Considering the null geodesic (25) we can see that the surface of singularity is at $r = 3\beta_0^{1/3}$ and the physical region is $0 \leq r < 3\beta_0^{1/3}$. Actually when $r \rightarrow 3\beta_0^{1/3}$ the observational relations break down as $z \rightarrow \infty$, $V \rightarrow \infty$, $d_l \rightarrow \infty$, $\rho \rightarrow \infty$, $\rho_v \rightarrow 0$.

As one can see the volume density vanishes at the big bang, a result that comes as a consequence of the definition adopted for ρ_v in equation (20). At the big bang singularity hypersurface the volume is infinite, but the total mass is finite. It is interesting to note that one of the postulates of the so-called ‘‘Pure Hierarchical Models’’ (Wertz 1970, p. 18) requires a vanishing global density as the distance goes infinite, and a similar result was also conjectured by Pietronero (1987) for a fractal distribution. This vanishing volume density in a spatially homogeneous model might be interpreted as meaning that the Einstein–de Sitter model is hierarchical at the asymptotic limit.

In order to investigate a possible fractal behaviour in this model, if we remember that $r \geq 0$, $d_l \geq 0$, equation (29) can be inverted to get

$$3\beta_0^{1/3} - r = 3\beta_0^{1/3} \left[\frac{1}{2} + \sqrt{\frac{d_l}{3\beta_0} + \frac{1}{4}} \right]^{-1}. \quad (37)$$

With the equation above we can express the number count (30) and the volume density (32) in terms of the luminosity distance. These expressions may be written as

$$N_c = \frac{6\beta_0}{9M_G} \left[1 - \left(\frac{1}{2} + \sqrt{\frac{d_l}{3\beta_0} + \frac{1}{4}} \right)^{-1} \right]^3, \quad (38)$$

$$\rho_v = \frac{1}{6\pi\beta_0^2} \left(\frac{1}{2} + \sqrt{\frac{d_l}{3\beta_0} + \frac{1}{4}} \right)^{-6}. \quad (39)$$

The power series expansions of equations (38) and (39) are

$$N_c = \frac{2d_l^3}{81M_G\beta_0^2} \left[1 - \frac{2}{\beta_0}d_l + \frac{3}{\beta_0^2}d_l^2 - \frac{110}{27\beta_0^3}d_l^3 + \dots \right], \quad (40)$$

$$\rho_v = \frac{1}{6\pi\beta_0^2} \left[1 - \frac{2}{\beta_0}d_l + \frac{3}{\beta_0^2}d_l^2 - \frac{110}{27\beta_0^3}d_l^3 + \dots \right]. \quad (41)$$

We can see again that ρ_v is constant at the origin and remains virtually unchanged for small values of d_l , a behaviour hardly in line with the de Vaucouleurs’ density power

law (22). By comparing equation (40) with equation (21) we can also see that at small values of the luminosity distance, the model has fractal dimension equal to 3, that is, corresponding to a homogeneous distribution, and this value changes as d_l increases. Both results are not in line with what one would expect if Einstein–de Sitter had a single fractal behaviour.

The results above can be seen graphically by plotting the logarithm of ρ_v against d_l . This method offers us an alternative way of investigating whether Einstein–de Sitter has any fractal behaviour. One cannot carry out this graphic investigation by means of equation (21) because not only it does already assume a single fractal behaviour for the dust, which is what one wants to test, but also it involves the unknown constant σ . What one can say is that if the distribution remains homogeneous throughout the null geodesic, the volume density will not change and, therefore, $\log \rho_v = \text{constant}$ and $D = 3$. In this homogeneous situation the plot of $\log \rho_v$ against $\log d_l$ would consist of a straight line with zero slope.

Figure 1 shows the log-log graph of ρ_v plotted against d_l using their parametric relations (32) and (29) for $0.001 \leq r \leq 1.5$. One can clearly see that for the luminosity distance range $-1.2 \leq \log d_l \leq -1$ ($0.06 \lesssim d_l \lesssim 0.1$) the distribution starts to deviate significantly from a homogeneous distribution. As at small values of d_l the distribution is still homogeneous, we can interpret the deviation as due to a significant distancing from the initial time hypersurface. For $\log d_l \geq -0.8$ ($d_l \gtrsim 0.16$) on the observational relations are being evaluated at increasingly very different and earlier epochs. The graph clearly shows that the Einstein–de Sitter model does not produce the results expected from a fractal model.

It is also interesting to note that $\log d_l = -0.8$ ($r \cong 0.08$) corresponds to $z \approx 0.04$ and that the deepest inhomogeneous structures identified in the IRAS survey are at $z \approx 0.07$ (Saunders et al. 1991). Therefore, for the range where Einstein–de Sitter predicts homogeneity, the IRAS survey does not find it and for the regions beyond the model starts to deviate from it. If we accept these results at their face value they might bring difficulties for a Einstein–de Sitter universe as inhomogeneities are certainly present at least within $z \approx 0.07$.

Recently, Efstathiou et al. (1991) have studied a deep survey of faint blue galaxies with magnitudes around 24–26. It would be of interest to see how their sample and their modelling relates to the model of this work. For this purpose, let us first find the redshift distribution of the sources in this model. Considering equation (28) we can write the number counts (30) along the geodesic as a function of the redshift

of the sources:

$$N_c = \frac{6\beta_0}{M_G} \left[1 - \frac{1}{\sqrt{1+z}} \right]^3. \quad (42)$$

The number of objects per unit redshift interval may be written as

$$\frac{dN_c}{dz} = \frac{9\beta_0}{M_G} \frac{1}{(1+z)^{3/2}} \left[1 - \frac{1}{\sqrt{1+z}} \right]^2. \quad (43)$$

It is easy to see that equation (43) starts increasing from the origin and then, after peaking at $z \cong 1.78$, begins to decrease steadily, without a sharp fall.

Although it is not obvious how equation (43) relates to the data of Efstathiou et al. (1991), especially because of the uncertainty of the redshift distribution for their faint galaxies, we can at least say that one of the redshift distributions which they considered “realistic” has some functional similarities to equation (43), in the sense that it also starts increasing and then decreases steadily, without sharp cutoffs, after a maximum. Nonetheless, we have to bear in mind that the calculations shown here assume that the sources have uniform intrinsic luminosity, and that means that specific redshifts correspond to specific luminosity distances and apparent magnitudes. This is certainly a simplification as in real astronomical observations one has objects of different magnitudes at equivalent redshifts, and vice-versa. To be able to infer the objects’ distance from apparent magnitudes, one needs a model based on their intrinsic physical processes that gives intrinsic luminosity. In other words, one must have a model for galactic evolution or, perhaps more specifically, galactic color evolution. We shall not consider here the problem of linking equation (43) to real astronomical data acquisition methods and galactic evolution models.

Before the end of this section, let us briefly discuss the claim of Efstathiou et al. (1991), based on calculations of the two-point angular correlation function, that faint blue galaxies are weakly clustered. The important point is that if the galactic distribution forms a fractal structure, it would not necessarily be easily detected by the angular correlation function. This point was highlighted by Coleman & Pietronero (1991) who measured angular correlations of simulations of three dimensional fractal structures and found out that they have a tendency to homogenize at angles which approach a fraction of the sample angle. This is explained by the peculiar properties of fractals whose dimensions may change once they are projected. Thus, it would appear that conclusions about homogeneity based on the current angular correlation analysis are open to controversy. Fractals are basically simple, but they are also subtle and therefore can be elusive.

4 Conclusion

In this work the theory developed in Ribeiro (1992) for a relativistic hierarchical (fractal) cosmology has been applied to the special case of the Einstein–de Sitter model. We have calculated the observational relations for this particular model and studied whether it is compatible with a fractal description of galactic clustering. The results show that the model under consideration does not possess fractal features along the backward null cone and fails to predict the inhomogeneities observed at the scale of the IRAS survey. Besides, although the model is homogeneous, it does not appear to remain so along the past light cone. We have also found a vanishing volume density at the big bang. Finally, we have obtained the equation for redshift distribution of the sources in a Einstein-de Sitter model and discussed the possible relationship of this theoretical distribution with the clustering analysis of a recent survey of deep faint galaxies.

The inhomogeneity of the Einstein-de Sitter model is a consequence of the manner densities are expressed here. The volume density ρ_v is measured along the past null geodesic which goes through hypersurfaces of t constant, where each one has different values for the proper density. It changes along the geodesic because it is a cumulative density, averaging at bigger and bigger volumes, in a way that adds more and more different local densities of each spatial section of the model. The proper density ρ also changes when expressed along the geodesic and for the same physical reason. Hence, in this sense the “homogeneity” of the Einstein-de Sitter model does not survive.

The analysis shown above is an example of the problems of applying measures of “homogeneity” even to a universe model that really is spatially homogeneous. It is clear that a “homogeneous” model may be taken to be inhomogeneous, depending on how we look at it. Furthermore, as the model apparently has difficulties in accounting for the observed inhomogeneities, that may compel us to conjecture that the Einstein–de Sitter model could end up being a victim of its own strength: it is too simple. From a relativistic fractal cosmological point of view what is really needed are solutions of Einstein’s field equations with fractal behaviour along the past null cone. We intend to deal with such solutions in paper III.

Acknowledgements

It is my pleasure to thank W. B. Bonnor for discussions and many suggestions which led to this paper and enriched it. I am also grateful to M. A. H. MacCallum for numerous valuable discussions and suggestions, S. Oliver and P. Coles for discussions and the referee for helpful comments. This work had the financial support of Brazil's Ministry of Education agency CAPES.

References

- Bonnor, W. B. 1972, M. N. R. A. S., 159, 261.
- Coleman, P. H., & Pietronero, L. 1991, preprint.
- Efstathiou, G. et al. 1991, Ap. J. (Letters), 380, L47.
- Jones, B. J. T., et al. 1988, Ap. J. (Letters), 332, L1.
- de Lapparent, V., Geller, M. J., & Huchra, J. P. 1986, Ap. J. (Letters), 302, L1.
- Martínez, V. J., et al. 1990, Ap. J., 357, 50.
- Pietronero, L. 1987, Physica, 144A, 257.
- Ribeiro, M. B. 1992, Ap. J., 388, 1 (paper I).
- Saunders, W., et al. 1991, Nature, 349, 32.
- Tolman, R. C. 1934, Proc. Nat. Acad. Sci. (Wash.), 20, 169.
- Wertz, J. R. 1970, "Newtonian Hierarchical Cosmology", PhD thesis (University of Texas at Austin).

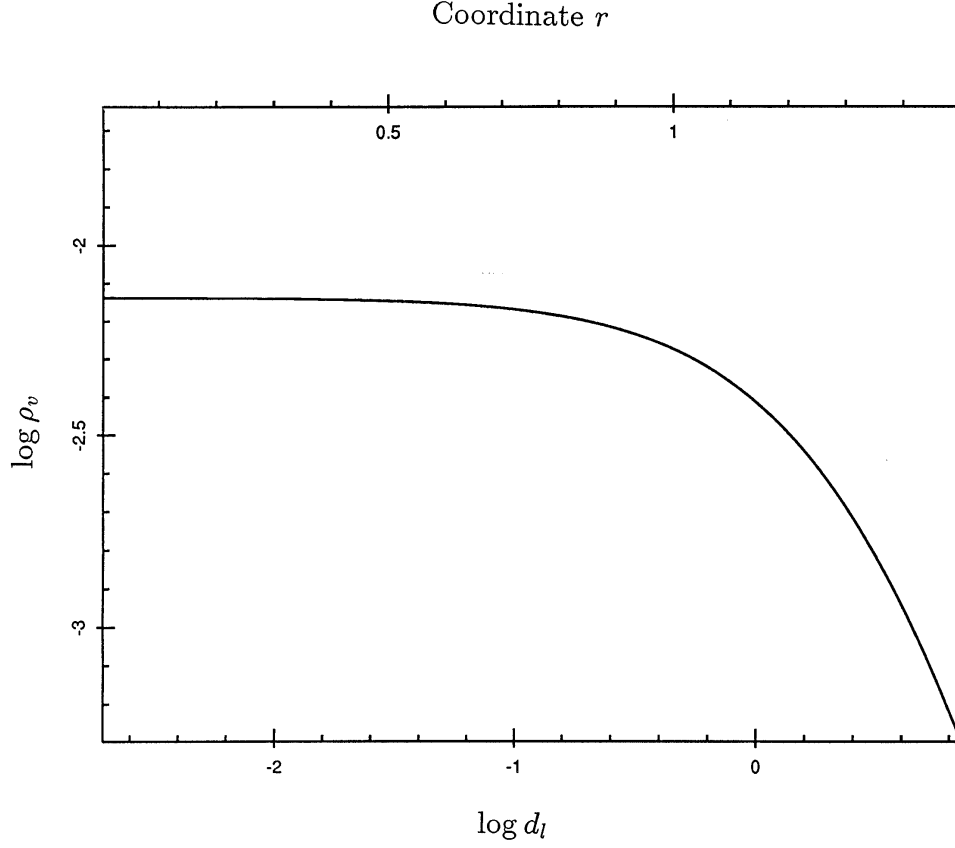


Figure 1

Figure Caption

Log-log plot of volume density ρ_v given by equation (32) against the luminosity distance d_l given by equation (29) in the range $0.001 \leq r \leq 1.5$ and with $\beta_0 = 2.7$ (units are geometrical with $c = G = 1$, distance is given in Gpc and the Hubble constant assumed to be 75 km/s/Mpc). We can clearly see that in the Einstein–de Sitter model the distribution does not appear to remain homogeneous along the geodesic and the fractal dimension departs from the initial value 3.